Economics Letters 58 (1998) 17–29

economics letters

Generalized impulse response analysis in linear multivariate models

H. Hashem Pesaran[a,*], Yongcheol Shin[b]

[a]*Trinity College, Cambridge, UK*
[b]*Department of Applied Economics, University of Cambridge, Cambridge, UK*

Received 29 May 1997; received in revised form 17 June 1997; accepted 3 July 1997

Abstract

Building on Koop, [Koop et al. (1996) Impulse response analysis in nonlinear multivariate models. Journal of Econometrics 74, 119–147] we propose the 'generalized' impulse response analysis for unrestricted vector autoregressive (VAR) and cointegrated VAR models. Unlike the traditional impulse response analysis, our approach does not require orthogonalization of shocks and is invariant to the ordering of the variables in the VAR. The approach is also used in the construction of order-invariant forecast error variance decompositions. © 1998 Elsevier Science S.A.

Keywords: Generalized impulse responses; Forecast error variance decompositions; VAR; Cointegration

JEL classification: C13; C32; C51

1. Introduction

Following Sims (1980) seminal paper, dynamic analysis of vector autoregressive (VAR) models is routinely carried out using the 'orthogonalized' impulse responses, where the underlying shocks to the VAR model are orthogonalized using the Cholesky decomposition before impulse responses, or forecast error variance decompositions are computed. This approach is not, however, invariant to the ordering of the variables in the VAR. See, for example, Lütkepohl (1991), (Section 2.3.2).

In this note, building on Koop et al. (1996), we propose an alternative approach to impulse response which does not have the above shortcoming. We refer to this as the generalized impulse response analysis. In particular, we show that for a non-diagonal error variance matrix the orthogonalized and the generalized impulse responses coincide only in the case of the impulse responses of the shocks to the first equation in the VAR. In Section 4 the proposed approach is applied to the cointegrated VAR models, and it is shown that the maximum likelihood estimator of the generalized impulse responses is $\sqrt{T}$-consistent and asymptotically normally distributed.

We provide an empirical illustration of the substantial differences that could exist between the two

*Corresponding author. Address for correspondence: Faculty of Economics and Politics, University of Cambridge, Austin Robinson Building, Sidgwick Avenue, Cambridge CB3 9DD, UK. Tel.: +44 1223 335 216; fax: +44 1223 335 471; e-mail: MHP1@econ.cam.ac.uk

0165-1765/98/$19.00 © 1998 Elsevier Science S.A. All rights reserved.
PII S0165-1765(97)00214-0

approaches, using a trivariate VAR model containing U.S. quarterly observations on real investment and consumption expenditures and output over 1948(1)–1988(4).

2. The generalized impulse response functions

Consider the *augmented* vector autoregressive model,

$$x_t = \sum_{i=1}^{p} \boldsymbol{\Phi}_i \mathbf{x}_{t-i} + \Psi \mathbf{w}_t + \varepsilon_t, \quad t = 1, 2, \ldots, T, \tag{1}$$

where $\mathbf{x}_t = (x_{1t}, x_{2t}, \ldots, x_{mt})'$ is an $m \times 1$ vector of jointly determined dependent variables, $\mathbf{w}_t$ is an $q \times 1$ vector of deterministic and/or exogenous variables, and $\{\boldsymbol{\Phi}_i, i=1, 2, \ldots, p\}$ and $\boldsymbol{\Psi}$ are $m \times m$ and $m \times q$ coefficient matrices. We make the following standard assumptions: (see, for example, Lütkepohl, 1991, Chapter 2, and Pesaran and Pesaran, 1997, Section 19.3).

Assumption 2.1 $E(\varepsilon_t) = \mathbf{0}$, $E(\varepsilon_t \varepsilon_t{}') = \Sigma$ *for all t, where* $\Sigma = \{\sigma_{ij}, i, j=1, 2, \ldots, m\}$ *is an* $m \times m$ *positive definite matrix,* $E(\varepsilon_t \varepsilon_{t'}') = \mathbf{0}$ *for all* $t = t'$ *and* $E(\varepsilon_t | \mathbf{w}_t) = \mathbf{0}$.

Assumption 2.2 *All the roots of* $|\mathbf{I}_m - \sum_{i=1}^{p} \boldsymbol{\Phi}_i z^i| = 0$ *fall outside the unit circle.*

Assumption 2.3 $\mathbf{x}_{t-1}, \mathbf{x}_{t-2}, \ldots, \mathbf{x}_{t-\mathrm{p}}, \mathbf{w}_t$, $t=1, 2, \ldots, T$, *are not perfectly collinear.*

Under Assumption 2.2, $\mathbf{x}_t$ would be covariance-stationary, and (1) can be rewritten as the infinite moving average representation,

$$\mathbf{x}_t = \sum_{i=0}^{\infty} \mathbf{A}_i \varepsilon_{t-i} + \sum_{i=0}^{\infty} \mathbf{G}_i \mathbf{w}_{t-i}, \quad t = 1, 2, \ldots, T, \tag{2}$$

where the $m \times m$ coefficient matrices $\mathbf{A}_i$ can be obtained using the following recursive relations:

$$\mathbf{A}_i = \boldsymbol{\Phi}_1 \mathbf{A}_{i-1} + \boldsymbol{\Phi}_2 \mathbf{A}_{i-2} + \cdots + \boldsymbol{\Phi}_p \mathbf{A}_{i-p}, \quad i = 1, 2, \ldots, \tag{3}$$

with $\mathbf{A}_0 = \mathbf{I}_m$ and $\mathbf{A}_i = 0$ for $i < 0$, and $\mathbf{G}_i = \mathbf{A}_i \psi$.

An impulse response function measures the time profile of the effect of shocks at a given point in time on the (expected) future values of variables in a dynamical system. The best way to describe an impulse response is to view it as the outcome of a *conceptual* experiment in which the time profile of the effect of a hypothetical $m \times 1$ vector of shocks of size $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_m)'$, say, hitting the economy at time t is compared with a base-line profile at time $t+n$, given the economy's history. There are three main issues: (i) The types of shocks hitting the economy at time t; (ii) the state of the economy at time $t-1$ before being shocked; and (iii) the types of shocks expected to hit the economy from $t+1$ to $t+n$.

Denoting the known history of the economy up to time $t-1$ by the non-decreasing information set $\boldsymbol{\Omega}_{t-1}$, the generalized impulse response function of $\mathbf{x}_t$ at horizon n, advanced in Koop et al. (1996), is defined by

$$\boldsymbol{GI}_x(n, \boldsymbol{\delta}, \boldsymbol{\Omega}_{t-1}) = E(\boldsymbol{x}_{t+n}|\varepsilon_t = \delta, \boldsymbol{\Omega}_{t-1}) - E(\boldsymbol{x}_{t+n}|\boldsymbol{\Omega}_{t-1}). \tag{4}$$

Using (4) in (2), we have $\boldsymbol{GI}_x(n, \boldsymbol{\delta}, \boldsymbol{\Omega}_{t-1}) = \boldsymbol{A}_n\boldsymbol{\delta}$, which is independent of $\boldsymbol{\Omega}_{t-1}$, but depends on the composition of shocks defined by $\boldsymbol{\delta}$.[1]

Clearly, the appropriate choice of hypothesized vector of shocks, $\boldsymbol{\delta}$, is central to the properties of the impulse response function. The traditional approach, suggested by Sims (1980), is to resolve the problem surrounding the choice of $\boldsymbol{\delta}$ by using the Cholesky decomposition of $\boldsymbol{\Sigma}$:

$$\boldsymbol{PP}' = \boldsymbol{\Sigma}, \tag{5}$$

where $\boldsymbol{P}$ is an $m \times m$ lower triangular matrix. Then, (2) can be rewritten as

$$\boldsymbol{x}_t = \sum_{i=0}^{\infty} (\boldsymbol{A}_i\boldsymbol{P})(\boldsymbol{P}^{-1}\varepsilon_{t-i}) + \sum_{i=0}^{\infty} G_i\boldsymbol{w}_{t-i} = \sum_{i=0}^{\infty} (\boldsymbol{A}_i\boldsymbol{P})\boldsymbol{\xi}_{t-i} + \sum_{i=0}^{\infty} \boldsymbol{G}_i\boldsymbol{w}_{t-i}, \quad t = 1, 2, \ldots, T, \tag{6}$$

such that $\boldsymbol{\xi}_t = \boldsymbol{P}^{-1}\varepsilon_t$ are orthogonalized; namely, $E(\boldsymbol{\xi}_t\boldsymbol{\xi}_t') = \boldsymbol{I}_m$. Hence, the $m \times 1$ vector of the orthogonalized impulse response function of a unit shock to the jth equation on $\boldsymbol{x}_{t+n}$ is given by

$$\psi_j^o(n) = \boldsymbol{A}_n\boldsymbol{P}\boldsymbol{e}_j, \quad n = 0, 1, 2, \ldots, \tag{7}$$

where $\boldsymbol{e}_j$ is an $m \times 1$ selection vector with unity as its jth element and zeros elsewhere.

An alternative approach would be to use (4) directly, but instead of shocking all the elements of ε_t, we could choose to shock only one element, say its jth element, and integrate out the effects of other shocks using an assumed or the historically observed distribution of the errors. In this case we have

$$\boldsymbol{GI}_x(n, \delta_j, \boldsymbol{\Omega}_{t-1}) = E(\boldsymbol{x}_{t+n}|\varepsilon_{jt} = \delta_j, \boldsymbol{\Omega}_{t-1}) - E(\boldsymbol{x}_{t+n}|\boldsymbol{\Omega}_{t-1}). \tag{8}$$

Assuming that ε_t has a multivariate normal distribution, it is now easily seen (see also Koop et al. (1996)) that[2]

$$E(\varepsilon_t|\varepsilon_{jt} = \delta_j) = (\sigma_{1j}, \sigma_{2j}, \ldots \sigma_{mj})'\sigma_{jj}^{-1}\delta_j = \boldsymbol{\Sigma}\boldsymbol{e}_j\sigma_{jj}^{-1}\delta_j.$$

Hence, the $m \times 1$ vector of the (unscaled) generalized impulse response of the effect of a shock in the jth equation at time t on x_{t+n} is given by

$$\left(\frac{\boldsymbol{A}_n\boldsymbol{\Sigma}\boldsymbol{e}_j}{\sqrt{\sigma_{jj}}}\right)\left(\frac{\delta_j}{\sqrt{\sigma_{jj}}}\right), \quad n = 0, 1, 2, \ldots. \tag{9}$$

By setting $\delta_j = \sqrt{\sigma_{jj}}$, we obtain the scaled generalized impulse response function by

$$\psi_j^g(n) = \sigma_{jj}^{-\frac{1}{2}}\boldsymbol{A}_n\boldsymbol{\Sigma}\boldsymbol{e}_j, \quad n = 0, 1, 2, \ldots, \tag{10}$$

[1]This history invariance property of the impulse response is specific to linear systems and does not carry over to non-linear models.

[2]When the distribution of the errors ε_t are non-normal, one could obtain the conditional expectations $E(\varepsilon_t|\varepsilon_{jt} = \boldsymbol{\delta}_j)$ by stochastic simulations, or by resampling techniques if the distribution of errors is not known.

which measures the effect of one standard error shock to the jth equation at time t on expected values of $\boldsymbol{x}$ at time $t+n$.

Finally, the above generalized impulses can also be used in the derivation of the forecast error variance decompositions, defined as the proportion of the n-step ahead forecast error variance of variable i which is accounted for by the innovations in variable j in the VAR. For an analysis of the forecast error variance decompositions based on the orthogonalized impulse responses see Lütkepohl, 1991, Section 2.3.3. Denoting the orthogonalized and the generalized forecast error variance decompositions by $\theta^{o}_{ij}(n)$ and $\theta^{g}_{ij}(n)$, respectively, then for $n=0, 1, 2, \ldots,$

$$\theta^{o}_{ij}(n) = \frac{\sum_{l=0}^{n}(\boldsymbol{e}_i'\boldsymbol{A}_l\boldsymbol{P}\boldsymbol{e}_j)^2}{\sum_{l=0}^{n}(\boldsymbol{e}_i'\boldsymbol{A}_l\boldsymbol{\Sigma}\boldsymbol{A}_l'\boldsymbol{e}_i)},\ \theta^{g}_{ij}(n) = \frac{\sigma_{ii}^{-1}\sum_{l=0}^{n}(\boldsymbol{e}_i'\boldsymbol{A}_l\boldsymbol{\Sigma}\boldsymbol{e}_j)^2}{\sum_{l=0}^{n}\boldsymbol{e}_i'\boldsymbol{A}_l\boldsymbol{\Sigma}\boldsymbol{A}_l'\boldsymbol{e}_i}, \quad i, j = 1, \ldots, m.$$

Notice that by construction $\sum_{j=1}^{m}\theta^{o}_{ij}(n)=1$. However, due to the non-zero covariance between the original (non-orthogonalized) shocks, in general $\sum_{j=1}^{m}\theta^{g}_{ij}(\mathrm{n})\neq 1$.[3]

3. The relationship between generalized and orthogonalized impulse responses

The orthogonalized and the generalized impulse response functions, $\psi_j^{o}(n)$ and $\psi_j^{g}(n)$, differ in a number of respects. The generalized impulse responses are invariant to the reordering of the variables in the VAR, but this is not the case with the orthogonalized ones. Typically there are many alternative reparametrizations that could be employed to compute orthogonalized impulse responses, and there is no clear guidance as to which one of these possible parameterizations should be used. In contrast, the generalized impulse responses are unique and fully take account of the historical patterns of correlations observed amongst the different shocks.

The relationship between the two impulse responses are set out in the following proposition:[4]

Proposition 3.1 *The generalized and the orthogonalized impulse responses coincide if Σ is diagonal. In the case where Σ is non-diagonal,*

$$\psi_j^{g}(n) \neq \psi_j^{o}(n) \text{ for } j = 2, 3, \ldots, m,$$

and the two impulse responses are the same only for $j=1$.

Proof. The first part of Proposition 3.1 holds trivially. Next, rewrite $\psi_j^{g}(n)$ and $\psi_j^{o}(n)$ as

$$\psi_j^{g}(n) = \boldsymbol{A}_n\varphi_j^{g},\ \psi_j^{o}(n) = \boldsymbol{A}_n\varphi_j^{o},$$

such that $\varphi_j^{g} = \sigma_{jj}^{-(1/2)}\boldsymbol{\Sigma}\boldsymbol{e}_j$ and $\varphi_j^{o} = \boldsymbol{P}\boldsymbol{e}_j$. Noticing that

[3]For a further discussion of the generalised forecast error variance decompositions see Pesaran and Pesaran, 1997, Section 19.5.

[4]Proposition 3.1 also applies to the relationship between the generalized and the orthogonalised forecast error decompositions.

$$\varphi_j^g = \sigma_{jj}^{-\frac{1}{2}}(\sigma_{1j}, \sigma_{2j}, \ldots, \sigma_{mj})', \text{ for } j = 1, 2, \ldots, m,$$

$$\varphi_1^o = (p_{11}, p_{21}, \ldots, p_{m1})', \ldots, \varphi_j^o = (0, \ldots, 0, p_{jj}, \ldots, p_{mj})', \ldots, \varphi_m^o = (0, \ldots, 0, p_{mm})',$$

it is then easily seen that

$$\varphi_j^g \neq \varphi_j^o \text{ and } \psi_j^g \neq \psi_j^o \text{ for } j = 2, \ldots, m.$$

When $j = 1$,

$$\varphi_1^g = \sigma_{11}^{-\frac{1}{2}}(\sigma_{11}, \sigma_{21}, \ldots, \sigma_{m1})'. \tag{11}$$

Using the equality $\boldsymbol{PP}' = \Sigma$, we have

$$p_{11}^2 = \sigma_{11}, (\sigma_{11}, \sigma_{21}, \ldots, \sigma_{m1})' = (p_{11}^2, p_{11}p_{21}, \ldots, p_{11}p_{m1})'. \tag{12}$$

Using (12) in (11), we obtain $\varphi_1^g = \varphi_1^o$ and $\psi_1^g = \psi_1^o$. ■

4. The generalized impulse response analysis in a cointegrated VAR model

In this section we extend the generalized impulse analysis to a cointegrated VAR model, also known as a vector error correction (VEC) model. The econometric issues surrounding the analysis of VEC model are discussed, for example, in Johansen (1995) and Pesaran et al. (1997). Here we provide only a brief outline of the estimation issues, focusing attention on the generalized impulse response functions.

To deal with unit roots and cointegration, we replace Assumption 2.2 by

Assumption 4.1. *The roots of* $|\boldsymbol{I}_m - \sum_{i=1}^{p} \boldsymbol{\Phi}_i z^i| = 0$ *satisfy* $|z| > 1$ or $z = 1$.

In this case (1) can be transformed into the VEC form:

$$\Delta \boldsymbol{x}_t = -\boldsymbol{\Pi x}_{t-1} + \sum_{i=1}^{p-1} \boldsymbol{\Gamma}_i \Delta x_{t-i} + \boldsymbol{\Pi \Lambda w}_t + \varepsilon_t, t = 1, 2, \ldots, T, \tag{13}$$

where $\boldsymbol{\Pi} = \boldsymbol{I}_m - \sum_{i=1}^{p} \boldsymbol{\Phi}_i$, $\boldsymbol{\Gamma}_i = -\sum_{j=i+1}^{p} \boldsymbol{\Phi}_j$ for $i = 1, \ldots, p-1$, and $\boldsymbol{\Lambda}$ is an $m \times g$ matrix of unknown coefficients. The relationships between parameters in (1) and (13) can also be rewritten as

$$\boldsymbol{\Phi}_1 = \boldsymbol{I}_m - \boldsymbol{\Pi} + \boldsymbol{\Gamma}_1, \boldsymbol{\Phi}_i = \boldsymbol{\Gamma}_i - \boldsymbol{\Gamma}_{i-1} \text{ for } i = 2, \ldots, p-1, \boldsymbol{\Phi}_p = -\boldsymbol{\Gamma}_{p-1}. \tag{14}$$

Suppose that the system (13) is cointegrated in the sense that there exists an $m \times r$ matrix $\boldsymbol{\beta}$ such that the $r \times 1$ vector $\boldsymbol{z}_t = \boldsymbol{\beta}' \boldsymbol{x}_t$ is stationary, where $1 \leq r < m$. The cointegrating restrictions can be formally expressed as

$$\boldsymbol{\Pi} = \boldsymbol{\alpha \beta}', \tag{15}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $m \times r$ matrices of full rank r; that is, rank$(\boldsymbol{\Pi}) = r$. Finally, to ensure that the underlying variables in x_t are at most $I(1)$, we assume:

Assumption 4.2 *$\boldsymbol{\alpha}'_{\perp}\boldsymbol{\Gamma}\boldsymbol{\beta}_{\perp}$ has full rank, where $\boldsymbol{\Gamma}=\boldsymbol{I}_m-\sum_{i=i}^{p-1}\boldsymbol{\Gamma}_i$, and $\boldsymbol{\alpha}_{\perp}$ and $\boldsymbol{\beta}_{\perp}$ are $m\times(m-r)$ matrices of full column rank such that $\boldsymbol{\alpha}'\boldsymbol{\alpha}_{\perp}=\boldsymbol{0}$ and $\boldsymbol{\beta}'\boldsymbol{\beta}_{\perp}=\boldsymbol{0}$.*

Under Assumptions 4.1 and 4.2, and (15), $\boldsymbol{x}_t$ will be first-difference stationary, and therefore, $\Delta\boldsymbol{x}_t$ can be written as the infinite moving average representation (see, for example, Johansen (1995), (Chapter 4)),

$$\Delta\boldsymbol{x}_t=\sum_{i=0}^{\infty}\boldsymbol{C}_i\varepsilon_{t-i}+\sum_{i=0}^{\infty}\boldsymbol{C}_i\boldsymbol{\Pi}\boldsymbol{\Lambda}\boldsymbol{w}_{t-i},\quad t=1,2,\ldots,T. \tag{16}$$

Applying the definition of the scaled generalized impulse responses (see (10)) to (16), we obtain

$$\psi^{g}_{\Delta x,j}(n)=\sigma_{jj}^{-\frac{1}{2}}\boldsymbol{C}_n\boldsymbol{\Sigma}\boldsymbol{e}_j,\quad n=0,1,2,\ldots, \tag{17}$$

which measures the effect of the shock to the jth equation in (16) on $\Delta\boldsymbol{x}_{t+n}$. We also obtain the generalized impulse response functions of $\boldsymbol{x}_{t+n}$ with respect to a shock in the jth equation by

$$\psi^{g}_{x,j}(n)=\sigma_{jj}^{-\frac{1}{2}}\boldsymbol{B}_n\boldsymbol{\Sigma}\boldsymbol{e}_j,\quad n=0,1,2,\ldots, \tag{18}$$

where $\boldsymbol{B}_n=\sum_{j=0}^{n}\boldsymbol{C}_j$ is the 'cumulative effect' matrix with $\boldsymbol{B}_0=\boldsymbol{C}_0=\boldsymbol{I}_m$.

A necessary and sufficient condition for cointegration is given by $\boldsymbol{\beta}'\boldsymbol{C}(1)=0$, where $\boldsymbol{C}(1)=\sum_{i=0}^{\infty}\boldsymbol{C}_i$ with rank $[\boldsymbol{C}(1)]=m-r$ [see Engle and Granger (1987)]. Then, the cointegrating relations can be written as

$$\boldsymbol{z}_t=\boldsymbol{\beta}'\boldsymbol{x}_t=\sum_{i=0}^{\infty}\boldsymbol{\beta}'\boldsymbol{B}_i\varepsilon_{t-i}+\sum_{i=0}^{\infty}\boldsymbol{\beta}'\boldsymbol{B}_i\boldsymbol{\Pi}\boldsymbol{\Lambda}\boldsymbol{w}_{t-i}. \tag{19}$$

Hence, the generalized impulse response functions of $\boldsymbol{z}_t$ with respect to shock in the jth equation is given by

$$\psi^{g}_{z,j}(n)=\sigma_{jj}^{-\frac{1}{2}}\beta'\boldsymbol{B}_n\boldsymbol{\Sigma}\boldsymbol{e}_j,\quad n=0,1,2,\ldots. \tag{20}$$

Similarly, the orthogonalized impulse response functions of $\boldsymbol{x}_t$ and $\boldsymbol{z}_t$ with respect to a variable-specific shock in the jth equation are given by

$$\psi^{o}_{x,j}(n)=\boldsymbol{B}_nP\boldsymbol{e}_j,\ \psi^{o}_{z,j}(n)=\boldsymbol{\beta}'\boldsymbol{B}_nP\boldsymbol{e}_j,\quad n=0,1,2,\ldots. \tag{21}$$

In the present case it is important to note that the parametric restrictions implied by the deficiency in the rank of $\boldsymbol{\Pi}$ is taken into account, and therefore, the effects of shocks on the individual variables will be persistent, but these effects eventually vanish on the cointegrating relations, $\boldsymbol{z}_t=\boldsymbol{\beta}'\boldsymbol{x}_t$.[5]

The matrices $\{\boldsymbol{B}_n,\ n=1,2,\ldots\}$ can be computed from the underlying VAR coefficient matrices $\{\boldsymbol{\Phi}_i,\ i=1,\ldots,p\}$ using the following recursive relations [see Pesaran and Shin (1996)]:

$$\boldsymbol{B}_n=\boldsymbol{\Phi}_1\boldsymbol{B}_{n-1}+\boldsymbol{\Phi}_2\boldsymbol{B}_{n-2}+\cdots+\boldsymbol{\Phi}_p\boldsymbol{B}_{n-p},\quad n=1,2,\ldots, \tag{22}$$

[5]Notice that $\boldsymbol{B}_{\infty}=\boldsymbol{C}(1)$, and therefore $\boldsymbol{\beta}'\boldsymbol{B}_{\infty}=0$.

where $\boldsymbol{B}_0 = \boldsymbol{I}_m$ and $\boldsymbol{B}_n = 0$ for $n < 0$.

The ML estimators obtained from (13) can be used to obtain the ML estimators of $\psi^{g}_{x,j}(n)$ and $\psi^{g}_{z,j}(n)$, which we denote by $\hat{\psi}^{g}_{x,j}(n)$ and $\hat{\psi}^{g}_{z,j}(n)$, respectively.[6] Moreover, as shown in Appendix A, for $n = 0, 1, 2, \ldots,$

$$\sqrt{T}[\hat{\psi}^{g}_{x,j}(n) - \psi^{g}_{x,j}(n)] \overset{a}{\sim} N\{0, \boldsymbol{\Sigma}_x(n, j)\}, \tag{23}$$

$$\sqrt{T}[\hat{\psi}^{g}_{z,j}(n) - \psi^{g}_{z,j}(n)] \overset{a}{\sim} N\{0, \boldsymbol{\Sigma}_z(n, j)\}, \tag{24}$$

where '$\overset{a}{\sim}$' denotes asymptotic equality in distribution, and $\Sigma_x(n, j)$ and $\Sigma_z(n, j)$ are given respectively by (A.15) and (A.17) in Appendix A.

5. An empirical illustration

In this section we illustrate our approach by estimating impulse response functions for the trivariate VAR model in investment, consumption and output previously analyzed by King et al. (1991) (KPSW) on the U.S. quarterly data. All three variables are in logarithms, measured on a per capita basis, and denoted by i, c, and y.

We first analyze the unrestricted VAR(4) model,

$$\boldsymbol{x}_t = \boldsymbol{a}_0 + \boldsymbol{a}_1 t + \sum_{j=1}^{4} \boldsymbol{\Phi}_j \boldsymbol{x}_{t-j} + \varepsilon_t, \tag{5.1}$$

where the ordering of the variables in $\boldsymbol{x}$ is chosen to be (i, c, y), $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ are 3×1 vectors of coefficients, $\boldsymbol{\Phi}_j$, $j = 1, 2, 3, 4$, are 3×3 matrices of coefficients, and t is running from 1948(1) to 1988(4).

After estimating the parameters in (5.1) consistently by OLS, we turn to estimate the orthogonalized and the generalized impulse response functions with respect to one standard error shock to the output equation using (7) and (10), respectively.[7] The results for the generalized and orthogonalized impulse responses are presented in Figs. 1 and 2, respectively. Fig. 1 shows that the output shocks have larger and more persistent effects on investment, followed by output then consumption. After over-shooting, the effects on output and consumption tend to die out after about 7 quarters. But, the responses of investment to the output shock shows a cyclical pattern over a relatively protracted period of time; its impact response is 1.5%, rising to 2.5% in the 2nd quarter, then declining sharply to −1% around the 10th quarter, and finally gradually tending towards 0.

A markedly different picture emerges from Fig. 2, which displays the orthogonalized impulse responses with output shock having much larger impacts on itself than on investment in the first few quarters after the shock. Notice also that by construction the impact effects of orthogonalized output

[6] For more details see Lütkepohl and Reimers (1992) and Pesaran and Shin (1996).

[7] Detailed estimation results are available upon request. All the computations reported in this section have been carried out using Microfit 4.0 [Pesaran and Pesaran (1997)].

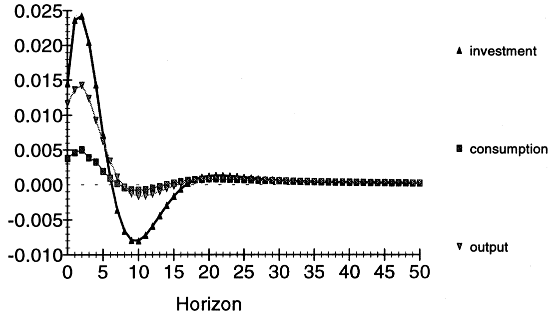

Fig. 1. Generalised impulse responses to one standard error shock in the output equation.

shocks on investment and consumption are zero. In general, the shape and the size of the two impulse responses are quite different.[8]

Next, following KPSW, we assume that all the three variables, are $I(1)$, and so analyze the three variables in the context of a cointegrated VAR(4) model with unrestricted intercepts and restricted-trend coefficients,[9]

$$\Delta \boldsymbol{x}_t = \boldsymbol{a}_0 + \boldsymbol{a}_1 t - \boldsymbol{\Pi x}_{t-1} + \sum_{j=1}^{3} \boldsymbol{\Gamma}_j \Delta \boldsymbol{x}_{t-j} + \varepsilon_t, \quad t = 1948(1) - 1988(4), \tag{5.2}$$

where $\boldsymbol{a}_1 = \boldsymbol{\Pi \gamma}$ and $\boldsymbol{\gamma}$ being 3×1 vector of unknown coefficients.

KPSW identify two cointegrating relations between the three variables, namely, $i-y$ and $c-y$, which are also referred to as 'great ratios'. With the two cointegrating relations one needs four restrictions (two per each cointegrating vector) to exactly identify them. Suppose that the four exactly-identifying restrictions are given by

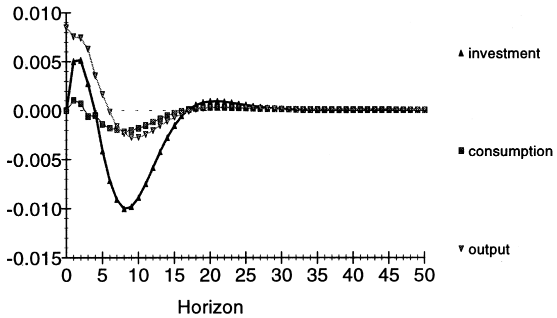

Fig. 2. Orthogonalised impulse responses to one standard error shock in the output equation.

[8]Clearly, the two impulse responses would have coincided if output was specified to be the first variable in the VAR. See Proposition 3.1.

[9]For more details on the choice of intercepts and trends in cointegrated VAR models see Pesaran and Pesaran (1997) and Pesaran et al. (1997).

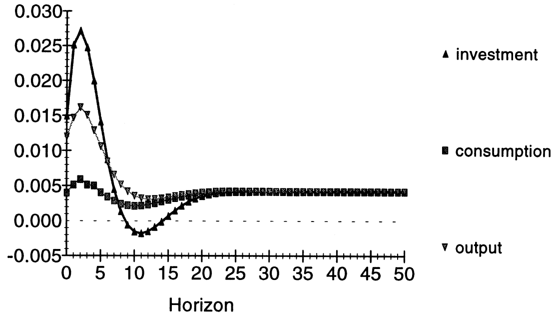

Fig. 3. Generalised impulse responses to one standard error shock in the error correction equation for output.

$$H_E: \begin{matrix} i \\ c \\ y \\ \text{trend} \end{matrix} \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \\ \beta_{41} & \beta_{42} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix}.$$

Under H_E, the ML estimates of the two cointegrating vectors obtained from (5.2) subject to the cointegrating restrictions, namely, rank$(\Pi)=2$, are as follows (see Pesaran and Shin (1997)):

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1.028\ (.27) & -1.046\ (.13) \\ .00017\ (.0011) & -.00008\ (.00054) \end{bmatrix}, \quad LL = 1552.1,$$

where the asymptotic standard errors are given in brackets, and LL is the maximized log-likelihood. Imposition of the full set of restrictions implied by two great ratios yields the maximized log-likelihood of 1548.8, resulting in the log-likelihood ratio statistic of 6.74, which is below the 95% critical value of the $\chi^2(4)$ distribution. Thus the 'great-ratio' hypothesis cannot be rejected, a finding which is in accordance with KPSW's conclusion.

Based on these two cointegrating relations, the generalized and orthogonalized impulse responses with respect to the output shocks can be estimated by (18), and (21), using the ML estimates. The time profiles of these impulse responses are displayed in Figs. 3 and 4, and give a very similar pattern to those obtained in the case of the unrestricted VAR model. A comparison of these figures further illustrates substantial differences that could exist between the two impulse responses.

Acknowledgements

We are grateful to James Mitchell and Ron Smith for helpful comments. Partial financial support from the ESRC (grant No. L116251016) and the Isaac Newton Trust of Trinity College, Cambridge, is gratefully acknowledged.

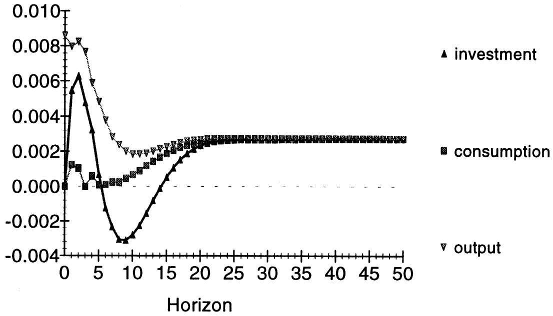

Fig. 4. Orthogonalised impulse responses to one standard error shock in the error correction equation for output.

Appendix A

Proofs of (23) and (24)

Let $\hat{\boldsymbol{\Gamma}}$, $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\Sigma}}$ be the maximum likelihood estimators of $\boldsymbol{\Gamma}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$ in the VEC model (13). Using the results in Lütkepohl and Reimers (1992) and Pesaran and Shin (1996), then the following asymptotic results (as $T \to \infty$) can be established:[10]

$$\sqrt{T} vec\{[\hat{\boldsymbol{\Gamma}}, -\hat{\boldsymbol{\alpha}}\hat{\beta}] - [\boldsymbol{\Gamma}, -\boldsymbol{\alpha}\beta]\} \overset{a}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_{CO}), \tag{A.1}$$

$$\sqrt{T} vec(\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}) \overset{a}{\sim} N\{0, 2\boldsymbol{P}_D(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\}, \tag{A.2}$$

where

$$\boldsymbol{\Sigma}_{CO} = (\boldsymbol{F}\boldsymbol{S}^{-1}\boldsymbol{F}') \otimes \boldsymbol{\Sigma}, \tag{A.3}$$

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{I}_{m(p-1)} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta} \end{bmatrix}, \quad \boldsymbol{S} = \operatorname*{Plim}_{T\to\infty} \begin{bmatrix} T^{-1}\boldsymbol{Y}\boldsymbol{Y}' & T^{-1}\boldsymbol{Y}\boldsymbol{X}'_{-1}\boldsymbol{\beta} \\ T^{-1}\boldsymbol{\beta}'\boldsymbol{X}_{-1}\boldsymbol{Y} & T^{-1}\boldsymbol{\beta}'\boldsymbol{X}_{-1}\boldsymbol{X}'_{-1}\boldsymbol{\beta} \end{bmatrix},$$

$\boldsymbol{Y}_t = [\Delta x'_{t-1}, \ldots, \Delta x'_{t-p+1}]'$, $\boldsymbol{Y} = [\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_T]$, $X_{-1} = [x_0, x_1, \ldots, x_{T-1}]$, $\boldsymbol{\Gamma} = [\boldsymbol{\Gamma}_1, \ldots, \boldsymbol{\Gamma}_{p-1}]$, and $\boldsymbol{P}_D = \boldsymbol{D}_m(\boldsymbol{D}'_m\boldsymbol{D}_m)^{-1}\boldsymbol{D}'_m$ is the projection matrix based on the duplication matrix, $\boldsymbol{D}_m$.Defining an $m^2p \times 1$ vector, $\phi = \text{vec}(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2, \ldots, \boldsymbol{\Phi}_p)$, and using (A.1), (A.2), (14) and (22), it is straightforward to show that

$$\sqrt{T} vec(\hat{\phi} - \phi) \overset{a}{\sim} N\{\mathbf{0}, \boldsymbol{\Sigma}_{\phi}\}, \tag{A.4}$$

$$\sqrt{T} vec(\hat{\boldsymbol{B}}_n - \boldsymbol{B}_n) \overset{a}{\sim} N\{\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{B}_n} = \boldsymbol{K}_n\boldsymbol{\Sigma}_{\phi}\boldsymbol{K}'_n\}, \quad n = 1, 2, \ldots, \tag{A.5}$$

[10] In the context of the VEC model (13), the ML estimators of the short-run parameters ($\boldsymbol{\Gamma}$ and $\boldsymbol{\alpha}$) and of the long-run parameters ($\boldsymbol{\beta}$) are $\sqrt{T}$-consistent and T-consistent, respectively. For a proof see, for example, Pesaran and Shin (1997).

where

$$\boldsymbol{\Sigma}_{\phi} = (\boldsymbol{W}'\boldsymbol{F}\boldsymbol{S}^{-1}\boldsymbol{F}'\boldsymbol{W}) \otimes \boldsymbol{\Sigma}, \boldsymbol{K}_n = \sum_{i=0}^{n-1} \boldsymbol{J}(\boldsymbol{\Phi}')^{n-1-i} \otimes \boldsymbol{B}_i, \tag{A.6}$$

$$\underset{mp\times mp}{\boldsymbol{W}} = \begin{bmatrix} \boldsymbol{I}_m & -\boldsymbol{I}_m & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_m & -\boldsymbol{I}_m & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_m & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I}_m & -\boldsymbol{I}_m \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix}, \quad \underset{mp\times mp}{\boldsymbol{\Phi}} = \begin{bmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \boldsymbol{\Phi}_3 & \cdots & \boldsymbol{\Phi}_{p-1} & \boldsymbol{\Phi}_p \\ \boldsymbol{I}_m & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_m & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I}_m & \boldsymbol{0} \end{bmatrix}$$

and $\boldsymbol{J} = (\boldsymbol{I}_m, \boldsymbol{0}, \ldots, \boldsymbol{0})$ is an $m \times mp$ matrix.

Based on these results, we now derive the asymptotic distribution of the estimator of generalized impulse responses. First, consider

$$\hat{\psi}^{g}_{x,j}(n) - \psi^{g}_{x,j}(n) = \frac{\hat{\boldsymbol{B}}_n \hat{\boldsymbol{\Sigma}} \boldsymbol{e}_j (\boldsymbol{e}_j' \boldsymbol{\Sigma} \boldsymbol{e}_j)^{\frac{1}{2}} - \boldsymbol{B}_n \boldsymbol{\Sigma} \boldsymbol{e}_j (\boldsymbol{e}_j' \hat{\boldsymbol{\Sigma}} \boldsymbol{e}_j)^{\frac{1}{2}}}{(\boldsymbol{e}_j' \hat{\boldsymbol{\Sigma}} \boldsymbol{e}_j)^{\frac{1}{2}} (\boldsymbol{e}_j' \boldsymbol{\Sigma} \boldsymbol{e}_j)^{\frac{1}{2}}} \equiv \frac{\boldsymbol{N}_G}{D_G}, \tag{A.7}$$

where $\boldsymbol{e}_j$ is an $m \times 1$ selection vector, $\sigma_{jj} = \boldsymbol{e}_j' \boldsymbol{\Sigma} \boldsymbol{e}_j$ and $\hat{\sigma}_{jj} = \boldsymbol{e}_j' \hat{\boldsymbol{\Sigma}} \boldsymbol{e}_j$. Notice also that

$$\boldsymbol{N}_G = (\hat{\boldsymbol{B}}_n \hat{\boldsymbol{\Sigma}} - \boldsymbol{B}_n \boldsymbol{\Sigma}) \boldsymbol{e}_j (\boldsymbol{e}_j' \boldsymbol{\Sigma} \boldsymbol{e}_j)^{\frac{1}{2}} + \boldsymbol{B}_n \boldsymbol{\Sigma} \boldsymbol{e}_j [(\boldsymbol{e}_j' \boldsymbol{\Sigma} \boldsymbol{e}_j)^{\frac{1}{2}} - (\boldsymbol{e}_j' \hat{\boldsymbol{\Sigma}} \boldsymbol{e}_j)^{\frac{1}{2}}], \tag{A.8}$$

$$\hat{\boldsymbol{B}}_n \hat{\boldsymbol{\Sigma}} - \boldsymbol{B}_n \boldsymbol{\Sigma} = (\hat{\boldsymbol{B}}_n - \boldsymbol{B}_n)(\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}) + \boldsymbol{B}_n (\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}) + (\hat{B}_n - \boldsymbol{B}_n) \boldsymbol{\Sigma}, \tag{A.9}$$

and by the Taylor series expansion

$$(\boldsymbol{e}_j' \hat{\boldsymbol{\Sigma}} \boldsymbol{e}_j)^{\frac{1}{2}} = (\boldsymbol{e}_j' \boldsymbol{\Sigma} \boldsymbol{e}_j)^{\frac{1}{2}} + \frac{1}{2} (\boldsymbol{e}_j' \boldsymbol{\Sigma} \boldsymbol{e}_j)^{-\frac{1}{2}} (\boldsymbol{e}_j' \otimes \boldsymbol{e}_j') \, vec(\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}) + R, \tag{A.10}$$

where R is a scalar remainder term which in view of the consistency of the ML estimators can be shown to be of $o_p(1)$ order. Using (A.9) and (A.10) in (A.8), we have

$$\begin{aligned} \boldsymbol{N}_G = {} & [(\hat{\boldsymbol{B}}_n - \boldsymbol{B}_n)(\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}) + \boldsymbol{B}_n (\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}) + (\hat{\boldsymbol{B}}_n - \boldsymbol{B}_n) \boldsymbol{\Sigma}] \boldsymbol{e}_j (\boldsymbol{e}_j' \boldsymbol{\Sigma} \boldsymbol{e}_j)^{\frac{1}{2}} \\ & - \boldsymbol{B}_n \boldsymbol{\Sigma} \boldsymbol{e}_j \left[\frac{1}{2} (\boldsymbol{e}_j' \boldsymbol{\Sigma} \boldsymbol{e}_j)^{-\frac{1}{2}} (\boldsymbol{e}_j' \otimes \boldsymbol{e}_j') \, vec(\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}) + R \right], \end{aligned} \tag{A.11}$$

and also

$$D_G = (\boldsymbol{e}_j' \boldsymbol{\Sigma} \boldsymbol{e}_j) + \frac{1}{2} (\boldsymbol{e}_j' \otimes \boldsymbol{e}_j') \, vec(\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}) + R^* = \sigma_{jj} + o_p(1), \tag{A.12}$$

where R^* is another scalar remainder term of $o_p(1)$ order. Multiplying (A.11) by $\sqrt{T}$ and vectorizing the result,

$$\sqrt{T}\boldsymbol{N}_G = [(\boldsymbol{e}_j'\boldsymbol{\Sigma}\otimes\boldsymbol{I}_m)\sqrt{T}vec(\hat{\boldsymbol{B}}_n-\boldsymbol{B}_n)+(\boldsymbol{e}_j'\otimes\boldsymbol{B}_n)\sqrt{T}vec(\hat{\boldsymbol{\Sigma}}-\boldsymbol{\Sigma})]\sigma_{jj}^{\frac{1}{2}}$$
$$-\frac{1}{2}\psi_{x,j}^{g}(n)(\boldsymbol{e}_j'\otimes\boldsymbol{e}_j')\sqrt{T}vec(\hat{\boldsymbol{\Sigma}}-\boldsymbol{\Sigma})+o_p(1). \tag{A.13}$$

Therefore, multiplying (A.7) by $\sqrt{T}$ and using (A.12) and (A.13), we have

$$\sqrt{T}[\hat{\psi}_{x,j}^{g}(n)-\psi_{x,j}^{g}(n)] =$$
$$\frac{\left[(\boldsymbol{e}_j'\boldsymbol{\Sigma}\otimes\boldsymbol{I}_m),(\boldsymbol{e}_j'\otimes\boldsymbol{B}_n)-\frac{1}{2}\sigma_{jj}^{-\frac{1}{2}}\psi_{x,j}^{g}(n)(\boldsymbol{e}_j'\otimes\boldsymbol{e}_j')\right]}{\sqrt{\sigma_{jj}}}\begin{bmatrix}\sqrt{T}vec(\hat{\boldsymbol{B}}_n-\boldsymbol{B}_n)\\ \sqrt{T}vec(\hat{\boldsymbol{\Sigma}}-\boldsymbol{\Sigma})\end{bmatrix}+o_p(1). \tag{A.14}$$

Notice that the asymptotic distributions of $\sqrt{T}(\hat{\boldsymbol{\Sigma}}-\boldsymbol{\Sigma})$ and $\sqrt{T}(\hat{B}_n-\boldsymbol{B}_n)$ are independent but normally distributed. Then, using (A.2) and (A.5) we obtain (23), and $\boldsymbol{\Sigma}_x(n, j)$ is given by

$$\boldsymbol{\Sigma}_x(n,j)=\frac{1}{\sigma_{jj}}\{\boldsymbol{V}_{1n}^{x}\boldsymbol{\Sigma}_{B_n}\boldsymbol{V}_{1n}^{x\prime}+\boldsymbol{V}_{2n}^{x}[2\boldsymbol{P}_D(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})]\boldsymbol{V}_{2n}^{x\prime}\}, \tag{A.15}$$

where

$$\boldsymbol{V}_{1n}^{x}=\boldsymbol{e}_j'\boldsymbol{\Sigma}\otimes\boldsymbol{I}_m,\quad \boldsymbol{V}_{2n}^{x}=(\boldsymbol{e}_j'\otimes\boldsymbol{B}_n)-\frac{1}{2}\sigma_{jj}^{-\frac{1}{2}}\psi_{x,j}^{g}(n)(\boldsymbol{e}_j'\otimes\boldsymbol{e}_j').$$

Similarly, we obtain

$$\sqrt{T}[\hat{\psi}_{z,j}^{g}(n)-\psi_{z,j}^{g}(n)] =$$
$$\frac{\left[(\boldsymbol{e}_j'\boldsymbol{\Sigma}\otimes\boldsymbol{\beta}'),(\boldsymbol{e}_j'\otimes\boldsymbol{\beta}'\boldsymbol{B}_n)-\frac{1}{2}\sigma_{jj}^{-\frac{1}{2}}\psi_{z,j}^{g}(n)(\boldsymbol{e}_j'\otimes\boldsymbol{e}_j')\right]}{\sqrt{\sigma_{jj}}}\begin{bmatrix}\sqrt{T}vec(\hat{\boldsymbol{B}}_n-\boldsymbol{B}_n)\\ \sqrt{T}vec(\hat{\boldsymbol{\Sigma}}-\boldsymbol{\Sigma})\end{bmatrix}+o_p(1). \tag{A.16}$$

and

$$\boldsymbol{\Sigma}_z(n,j)=\frac{1}{\sigma_{jj}}\{\boldsymbol{V}_{1n}^{z}\boldsymbol{\Sigma}_{B_n}\boldsymbol{V}_{1n}^{z\prime}+\boldsymbol{V}_{2n}^{z}[2\boldsymbol{P}_D(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})]\boldsymbol{V}_{2n}^{z\prime}\}, \tag{A.17}$$

where

$$\boldsymbol{V}_{1n}^{z}=\boldsymbol{e}_j'\boldsymbol{\Sigma}\otimes\boldsymbol{\beta}',\quad V_{2n}^{z}=(\boldsymbol{e}_j'\otimes\boldsymbol{\beta}'\boldsymbol{B}_n)-\frac{1}{2}\sigma_{jj}^{-\frac{1}{2}}\psi_{z,j}^{g}(n)(\boldsymbol{e}_j'\otimes\boldsymbol{e}_j').$$

This establishes (24). Notice that (A.16) is asymptotically valid irrespective of whether we use the true value of $\boldsymbol{\beta}$ or its T-consistent estimator, $\hat{\beta}$. (See Corollary 1 in Pesaran and Shin (1996)).

References

Engle, R.F., Granger, C.W.J., 1987. Cointegration and error correction representation: estimation and testing. Econometrica 55, 251–276.

Johansen, S., 1995. Likelihood-based inference in cointegrated vector autoregressive models. Oxford University Press, Oxford.

King, R.G., Plosser, C.I., Stock, J.H., Watson, M.W., 1991. Stochastic trends and economic fluctuations. American Economic Review 81, 819–840.

Koop, G., Pesaran, M.H., Potter, S.M., 1996. Impulse response analysis in nonlinear multivariate models. Journal of Econometrics 74, 119–147.

Lütkepohl, H., 1991. Introduction to multiple time series analysis. Springer-Verlag, Berlin.

Lütkepohl, H., Reimers, H.E., 1992. Impulse response analysis of cointegrated systems. Journal of Economic and Dynamic Controls 16, 53–78.

Pesaran, M.H., Pesaran, B., 1997. Working with Microfit 4.0: An interactive econometric software package (DOS and Windows versions). Oxford University Press, Oxford.

Pesaran, M.H., Shin, Y., 1997. Long-run structural modelling. Unpublished manuscript. University of Cambridge. (Internet: http\\www.econ.cam.ac.uk\faculty\pesaran\).

Pesaran, M.H., Shin, Y., 1996. Cointegration and speed of convergence to equilibrium. Journal of Econometrics 71, 117–143.

Pesaran, M.H., Shin Y., Smith, R.J., 1997. Structural analysis of vector autoregressive models with exogenous $I(1)$ variables. DAE Working Papers Amalgamated Series No. 9706, University of Cambridge. (Internet: http\\www.econ.cam.ac.uk\faculty\pesaran\).

Sims, C., 1980. Macroeconomics and Reality. Econometrica 48, 1–48.